Failure due to fatigue in fiber bundles and solids

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We consider first a homogeneous fiber bundle model where all the fibers have got the same stress threshold (σ_c) beyond which all fail simultaneously in absence of noise. At finite noise, the bundle acquires a fatigue behavior due to the noise-induced failure probability at any stress σ. We solve this dynamics of failure analytically and show that the average failure time τ of the bundle decreases exponentially as $\sigma \to \sigma_c$ from below and $\tau = 0$ for $\sigma \geqslant \sigma_c$. We also determine the avalanche size distribution during such failure and find a power law decay. We compare this fatigue behavior with that obtained phenomenologically for the nucleation of the Griffith cracks. Next we study numerically the fatigue behavior of random fiber bundles having simple distributions of individual fiber strengths, at stress σ less than the bundle's strength $\widetilde{\sigma_c}$ (beyond which it fails instantly). The average failure time τ is again seen to decrease exponentially as $\sigma \to \widetilde{\sigma_c}$ from below and the avalanche size distribution shows similar power law decay. These results are also in broad agreement with experimental observations on fatigue in solids. We believe, these observations regarding the failure time are useful for quantum breakdown phenomena in disordered systems.

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I. INTRODUCTION

If one puts a load or stress (σ) on a solid or applies a voltage across an electrical circuit, a strain in the solid or a current through the circuit develops, which grows linearly (Hooke's law or Ohm's law) with the stress or voltage. If the external load on the system increases beyond its threshold limit (σ_c), the system fails: stress σ drops to zero due to fracture of the solid. The same occurs when the voltage on the network exceeds its limit and the current drops to zero due to the fuse of the circuit. Similar failures occur in dielectric materials when the electric field across the sample exceeds beyond its limit, and dielectric breakdown sets in. These failures usually nucleate around the defects in the solid, and the failure behavior and its statistics therefore crucially depend on the disorder or impurity distribution within the sample. These (quasistatic) failure properties of disordered solids have been studied extensively in recent years [1].

The dynamics of these failures in such systems are quite intriguing and is being studied very intensively these days. The critical dynamics of failure and its universality class in the democratic (global load sharing) fiber bundle model [2] has been established very recently [3]. These dynamics of failure are intrinsic and induced by the successive stress redistributions due to the failure of weaker fibers. However, an important kind of dynamical failure due to fatigue [4] occurs in such disordered systems when the fibers have an effective probability to fail under any stress [5], or as the microcracks within the solid grow at the crack tips with time due to chemical diffusion in the atmosphere [4]. The system then fails under a stress less than its normal strength (σ_c) and the time of failure (τ) depends on the load applied on the sample: $\tau \neq 0$ for $\sigma < \sigma_c$ and $\tau \simeq 0$ for $\sigma \geqslant \sigma_c$.

Here, we study first a phenomenological theory of crack nucleation, following Griffith [4,6], at finite temperature (T) and estimate the average failure time τ at any stress σ less than σ_c. We then develop a simple model of fatigue failure in a democratic fiber bundle model containing identical fibers (having equal threshold strength σ_c), where the fibers have a finite noise-induced failure probability. We have derived analytically the failure time for the bundle as a function of applied stress (σ) and noise ($\tilde{T}$). This result for the model is compared with that obtained for the phenomenological theory of crack nucleation at finite temperature. It is also in broad agreement with some recent experimental observations on fatigue in disordered solids [4,7]. Next, we derive the avalanche size distribution in this fixed strength model analytically and find robust power law decay. The above analytic results have been confirmed through the numerical studies on the same model. Finally we consider random fiber bundles with simple, yet nontrivial, distributions of the fiber strengths. Our numerical results show that for all these fiber bundles, the average time to failure (τ) decreases exponentially as the stress level σ approaches bundle's strength $\tilde{\sigma}_c$ from below and the avalanche size distributions show similar power law decay. We also discuss the plausibility of this (noise-induced) failure in other similar situations. In particular, we consider the validity of our model in quantum breakdown phenomena [8]: for example, in dielectric breakdown where the microscopic failure of the dielectric grains acquire a finite probability at any electric field due to quantum tunneling. The failure time and its variation with the strength of the external field in such a quantum failure can give us an estimate of the tunneling frequencies involved.

II. TIME FOR FRACTURE IN THE GRIFFITH NUCLEATION MODEL

Griffith in 1920, equating the released elastic energy of a growing crack inside a solid with the energy of the newly

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created crack surfaces, came to a quantitative estimate of the fracture strength of a solid containing an already existing fixed geometry microcrack. Assuming the linear elasticity behavior up to the breaking point of a brittle solid, the released elastic energy becomes $E_{el}=(\sigma^2/Y)l_0^3$ for a three-dimensional elastic solid under stress σ and modulus of elasticity Y, containing a microcrack of length l_0. The corresponding surface energy $E_s=\phi l_0^2$, where ϕ denotes the (crack) surface energy density. Using the concept of energy balance, Griffith equated the differential increment in the elastic energy dE_{el} with the corresponding surface energy increment dE_s as the crack propagates a further length dl and obtained

$$\sigma_c=\frac{\Omega}{\sqrt{l_0}}, \quad \Omega=\sqrt{Y\phi} \tag{1}$$

for equilibrium extension of the crack. Here σ_c is the amount of stress for and above which the microcrack propagates in no time (or in a small time dependent on the sound velocity) and causes a macroscopic failure of the sample.

This quasistatic picture can be extended to fatigue behavior of crack propagation for $\sigma<\sigma_c$. At any stress σ less than σ_c, the cracks can still nucleate [6] for a further extension at any finite temperature T with a probability $\sim\exp[-E/k_BT]$ and, consequently, the sample fails within a failure time τ given by

$$\tau^{-1}\sim\exp[-E(l_0)/k_BT], \tag{2}$$

where

$$E(l_0)=\phi l_0^2-\left(\frac{\sigma^2}{Y}\right)l_0^3 \tag{3}$$

is the crack (of length l_0) nucleation energy. Here k_B is the Boltzman factor. One can therefore express Eq. (2) as

$$\tau\sim\exp\left[A\left(1-\frac{\sigma^2}{\sigma_c^2}\right)\right], \tag{4}$$

where (the dimensionless parameter) $A=l_0^3\sigma_c^2/(Yk_BT)$ and σ_c is given by Eq. (1). This immediately suggests that the failure time τ grows exponentially for $\sigma<\sigma_c$ and approaches infinity if the stress σ is much less than σ_c when the temperature T is small, whereas τ becomes vanishingly small as the stress σ exceeds σ_c.

III. FATIGUE IN A HOMOGENEOUS DEMOCRATIC FIBER BUNDLE MODEL

Fatigue in fiber bundle model was first studied by Coleman in 1958 [5]. Thermally activated failures of fiber have recently been considered and approximate fatigue behavior has been studied [9]. We consider here a very simple fiber bundle model with noise-induced activated failure, for which the dynamics can be analytically solved.

Let us consider a homogeneous bundle of N fibers under load $L(=N\sigma)$, each having identical failure strength σ_c. Without any noise ($\tilde{T}=0$), the model is trivial: the bundle does not fail (failure time τ is infinity) for stress $\sigma<\sigma_c$, but it fails immediately ($\tau=0$) for $\sigma\geqslant\sigma_c$. We now assume that each such fiber has a finite probability $P(\sigma,\tilde{T})$ of failure at any stress σ induced by a nonzero noise $\tilde{T}$:

$$P(\sigma,\tilde{T})=\begin{cases}\dfrac{\sigma}{\sigma_c}\exp\left[-\dfrac{1}{\tilde{T}}\left(\dfrac{\sigma_c}{\sigma}-1\right)\right], & 0\leqslant\sigma\leqslant\sigma_c\\ 1, & \sigma>\sigma_c.\end{cases} \tag{5}$$

As one can see, each fiber now has got a nonvanishing probability $P(\sigma,\tilde{T})$ to fail under a stress $\sigma<\sigma_c$ at any nonzero noise parameter $\tilde{T}$. It may be noted that [unlike T in Eq. (2) or Eq. (4)] $\tilde{T}$ is a dimensionless noise parameter. $P(\sigma,\tilde{T})$ increases as $\tilde{T}$ increases and for $\sigma\geqslant\sigma_c$, $P(\sigma,\tilde{T})=1$. Unlike at $\tilde{T}=0$, the bundle therefore fails at $\sigma<\sigma_c$ after a finite time τ. Here we assume each fiber to have a fixed threshold σ_c, while their breaking probability at any σ $(<\sigma_c)$ is due to noise-activated hopping over the barrier height $(\sigma_c-\sigma)$. This differs from the earlier model studies [5,9] where the load distribution is noise induced.

A. Failure time

At $\tilde{T}\neq 0$ and under any stress σ $(<\sigma_c)$, some fibers fail due to noise and the load gets shared among the surviving fibers, which, in turn, enhances their stress value, inducing further failure. Denoting the fraction of fibers that remain intact at time t by U_t, a discrete time recursion relation (see Ref. [3]) can be written as

$$U_{t+1}=U_t\left[1-P\left(\frac{\sigma}{U_t},\tilde{T}\right)\right], \tag{6}$$

where $\sigma/U_t=L/(NU_t)$ is the redistributed load per fiber among the NU_t surviving fibers at time t. In the continuum limit, we can write the above recursion relation in a differential form

$$-\frac{dU}{dt}=\frac{\sigma}{\sigma_c}\exp\left[-\frac{1}{\tilde{T}}\left(\frac{\sigma_c}{\sigma}U-1\right)\right], \tag{7}$$

giving

$$\tau=\int_0^\tau dt=\frac{\sigma_c}{\sigma}\exp\left(-\frac{1}{\tilde{T}}\right)\int_0^1\exp\left[\frac{1}{\tilde{T}}\left(\frac{\sigma_c}{\sigma}\right)U\right]dU \tag{8}$$

or

$$\tau=\tilde{T}\exp\left(-\frac{1}{\tilde{T}}\right)\left[\exp\left(\frac{\sigma_c}{\sigma\tilde{T}}\right)-1\right], \tag{9}$$

for $\sigma<\sigma_c$. For $\sigma\geqslant\sigma_c$, starting from $U_t=1$ at $t=0$, one gets $U_{t+1}=0$ from Eqs. (5) and (6), giving $\tau=0$.

For small $\tilde{T}$ and as $\sigma\to\sigma_c$, $\tau\simeq\tilde{T}\exp[(\sigma_c/\sigma-1)/\tilde{T}]$. This failure time τ therefore approaches infinity as $\tilde{T}\to 0$. For σ

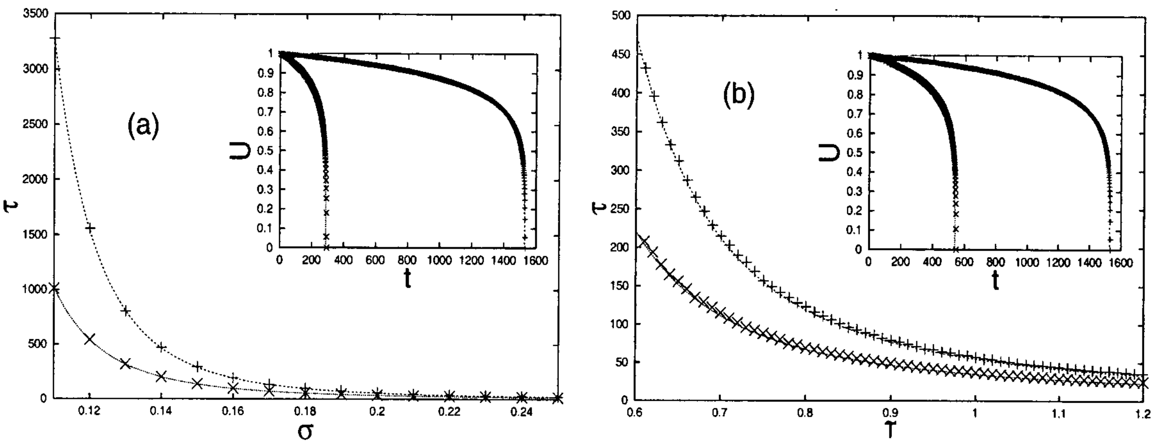

FIG. 1. The simulation results showing variation of average failure time τ against (a) stress σ and (b) noise $\tilde{T}$, for a bundle containing $N=10^5$ fibers. The theoretical results are shown by dotted and dashed lines [from Eq. (9)]. The insets show the simulation results for the variation of the fraction U of unbroken fibers with time t for different $\tilde{T}$ values [1.2 (cross) and 1.0 (plus)] in (a) and σ values [0.15 (cross) and 0.12 (plus)] in (b). The dotted and dashed lines represent the theoretical results [Eqs. (9) and (10)].

$<\sigma_c$, one gets finite failure time τ, which decreases exponentially as σ approaches σ_c or as $\tilde{T}$ increases and $\tau=0$ for $\sigma \geqslant \sigma_c$. This last feature is absent in the earlier formulations [9]. However, all these features are very desirable and are in qualitative agreement with the recent experimental observations [7]. This is also comparable with the phenomenological results from the Griffith theory discussed in the earlier section, although the crack size effect in the Griffith theory differes from that in the fiber bundle case. Our numerical study confirms the above analytic results [obtained using the continuum version of the recursion relation (6)] (see Fig. 1) well.

B. Avalanche size distribution

From the recursion relation (6) or (7), one can see that in each unit time interval a number of fibers break giving an avalanche size for the breaking. The avalanche size, therefore, is given by dU/dt and during the entire failure period τ, different sizes of avalanches take place. Solving for $U(t)$ from Eq. (7) one gets

$$U(t)=\frac{\sigma\tilde{T}}{\sigma_c}\ln\left[\frac{\tau-t}{\tilde{T}\,\exp(-1/\tilde{T})}+1\right], \qquad (10)$$

employing the expression (9) for τ. One can easily check that $U(t)=1$ at $t=0$ and $U(t)=0$ at $t=\tau$ (see Fig. 1). Also as $t\to t_c\equiv\tau$, $U(t)$ decays as $\ln(\tau-t)\sim(\tau-t)^{\beta}$ with $\beta=0_+$ from Eq. (10). Expressing dU/dt as the avalanche size m, one gets from Eq. (10)

$$m^{-1}\sim\frac{\tau-t}{\tilde{T}\,\exp(-1/\tilde{T})}+1\sim\tau-t, \qquad (11)$$

for $\tilde{T}\to 0$.

Here the avalanche size m can also be interpreted as the rate of breaking (dU/dt) and it varies with time as $(\tau-t)^{-\gamma}$, $\gamma=1$ as $t\to t_c\equiv\tau$. Since $\tau-t$ corresponds to the cumulative probability $\int_m^\infty D(m)dm$ of avalanches beyond t, one gets

$$D(m)\sim m^{-\alpha}, \quad \alpha=2, \qquad (12)$$

for the (differential) avalanche size distribution $D(m)$. Also, the exponent of power law decay in Eq. (12) is independent of stress σ and noise level $\tilde{T}$, which has been confirmed through numerical simulations (see Fig. 2). It may be mentioned that such avalanches manifest in the ultrasonic emissions during the propagation of fracture in the solid, and the ultrasonic amplitudes are also observed to have similar power law distribution [1].

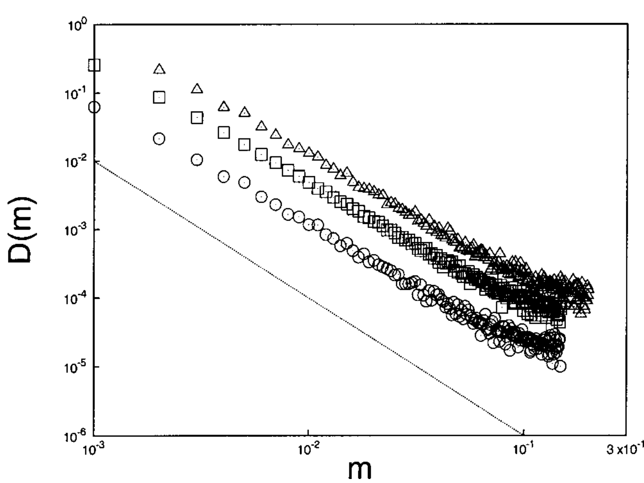

FIG. 2. The simulation results for the distribution $D(m)$ of avalanches in the bundle with $N=10^5$ (averaged over 10^3 realizations): $\sigma=0.2$, $\tilde{T}=0.8$ (triangle), $\sigma=0.15$, $\tilde{T}=0.8$ (circle), and $\sigma=0.15$, $\tilde{T}=1.0$ (square). The dashed line corresponds to a decay power 2.0.

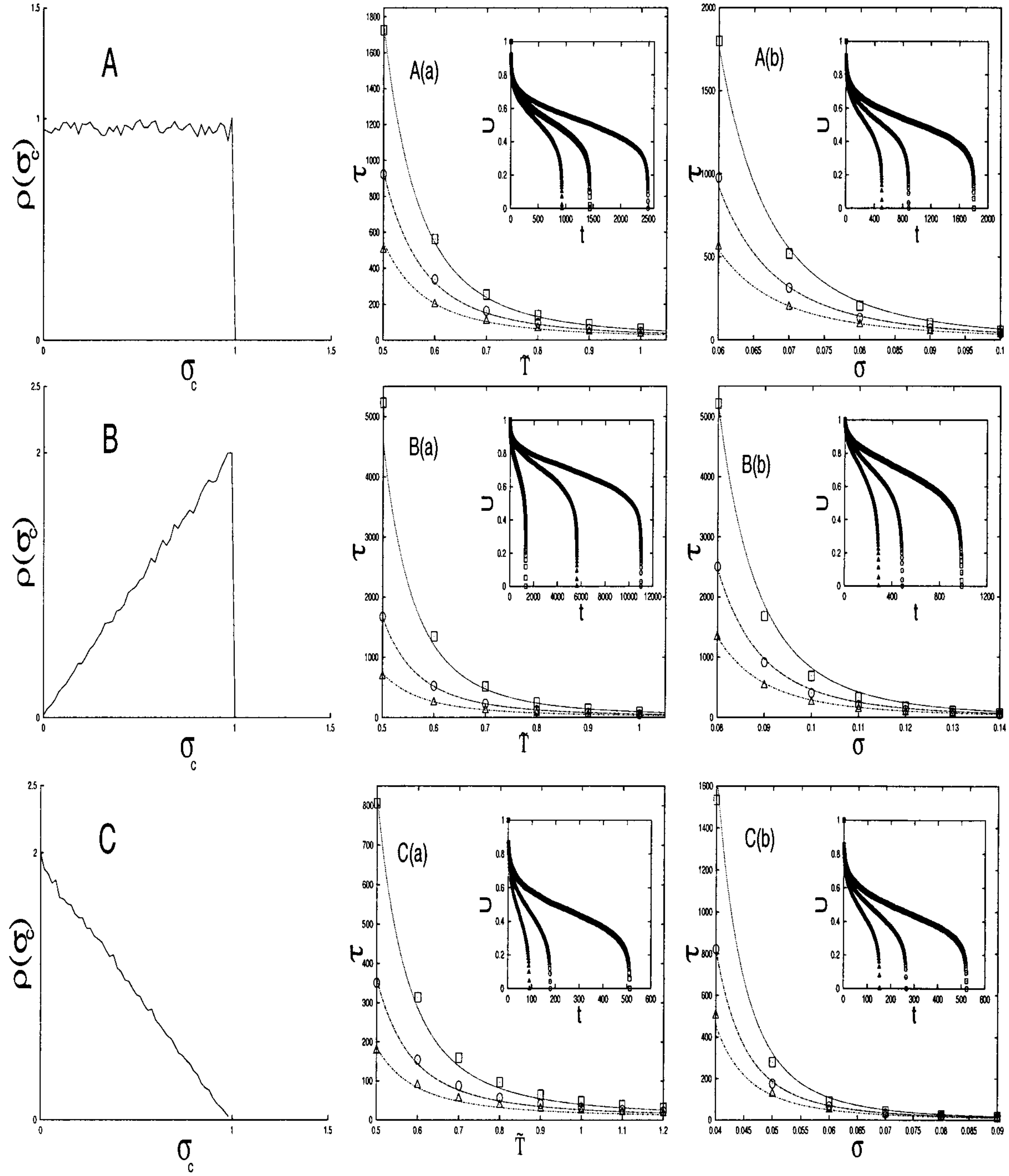

FIG. 3. Typical fiber strength distributions $\rho(\sigma_c)$ considered and the simulation results for fatigue behavior: (a) average failure time τ vs noise $\tilde{T}$ (for three different stress values σ) and (b) τ vs σ (for three different noise values $\tilde{T}$) are shown for $N=10^5$ fibers. The time variation of fraction of surviving fibers are shown in the insets for the three models: A with uniform $\rho(\sigma_c)$, B with linearly increasing $\rho(\sigma_c)$, and C with linearly decreasing $\rho(\sigma_c)$; all having a cutoff at $\sigma_c=1$. The dotted lines in (a) and (b) correspond to the fit with expression (13) where $\tilde{\sigma}_c \simeq 0.245$ in model A (exact value $=1/4$ [3]), $\tilde{\sigma}_c \simeq 0.370$ in model B (exact value $=\sqrt{4/27}$ [3]), $\tilde{\sigma}_c \simeq 0.148$ in model C (exact value $=4/27$ [3]).

IV. SIMULATION RESULTS FOR FATIGUE FAILURE IN RANDOM FIBER BUNDLE MODELS

In order to investigate the fatigue behavior in random fiber bundles we consider three different kinds of fiber strength distributions $\rho(\sigma_c)$: (1) uniform distribution of fiber strength where $\rho(\sigma_c)=1$ for $0<\sigma_c\leqslant 1$ and $\rho(\sigma_c)=0$ for $\sigma_c>1$, (2) linearly increasing distribution of fiber strength where $\rho(\sigma_c)=2\sigma_c$ for $0<\sigma_c\leqslant 1$ and $\rho(\sigma_c)=0$ for $\sigma_c>1$, and (3) linearly decreasing distribution of fiber strength where $\rho(\sigma_c)=2(1-\sigma_c)$ for $0<\sigma_c\leqslant 1$ and $\rho(\sigma_c)=0$ for $\sigma_c>1$. It has been already shown analytically [3], from the dynamics of failure in all these three kinds of fiber bundles in

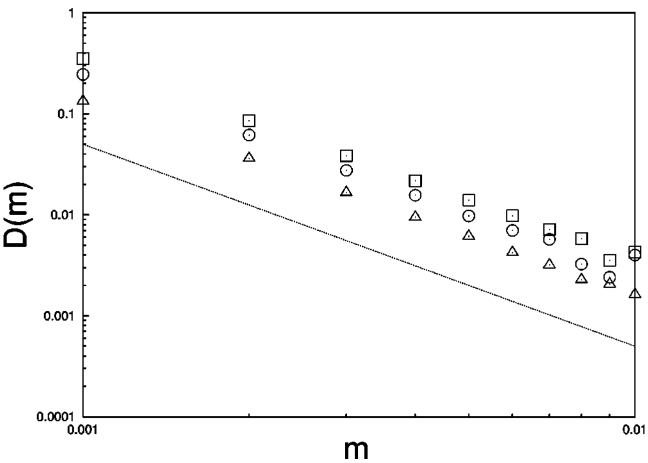

FIG. 4. The simulation results for the distributions $D(m)$ of avalanches (m) in the three random fiber bundles with $N=10^5$ (averaged over 4×10^3 realizations): for model A with $\sigma=0.07$, $\tilde{T}=0.5$ (square), for model B with $\sigma=0.12$, $\tilde{T}=0.4$ (circle), and for model C with $\sigma=0.04$, $\tilde{T}=0.5$ (triangle). The dashed line corresponds to a decay power 2.0.

the absence of any noise [vanishing T or $\tilde{T}$ in Eq. (5)], that the bundle's strength $\tilde{\sigma}_c=1/4$ for model A, $\tilde{\sigma}_c=\sqrt{4/27}$ for model B, and $\tilde{\sigma}_c=4/27$ for model C. We now consider the effect of the noise $\tilde{T}$ inducing the failure probability $P(\sigma,\tilde{T})=\exp[-(1/\tilde{T})(\sigma_c/\sigma-1)]$ for $0<\sigma\leqslant\sigma_c$ and 1 for $\sigma>\sigma_c$, in the (fatigue) dynamics of such bundles, where σ_c is the strength of the individual fibers in the bundle.

We have studied these numerically, using the Monte Carlo method (for bundles having $N=10^5$ or more fibers). We have considered bundles having the above three kinds (A, B, and C) of $\rho(\sigma_c)$ one by one. The noise-induced failure mentioned above is realized only in a Monte Carlo way. Taking averages typically over 10^3 Monte Carlo runs the fraction of unbroken fibers $U(t)$ at any time t at a fixed stress level $\sigma(<\tilde{\sigma}_c)$ is noted. At any σ, the average failure time τ [when $U(t)=0$] is extracted. The form of the distributions and the variations of average time with noise $\tilde{T}$ and stress σ are shown for the three types of bundles. We find that τ fits a form

$$\tau=\tilde{T}\exp\left(-\frac{1}{\tilde{T}}\right)\left[\exp\left(\frac{\tilde{\sigma}_c}{\sigma\tilde{T}}+\frac{1}{\tilde{T}}\right)-1\right] \tag{13}$$

for all types of bundles (indicated by dotted lines in Fig. 3). We find that this phenomenological form (13) is indeed very close to the analytic result (9) for the fixed strength fiber bundle; it is somewhat approximate for these bundles and fits better for lower noise ($\tilde{T}$) and stress (σ) levels. The avalanche size distributions in all these three models (A, B, and C) have been studied numerically (see Fig. 4) and we find them to follow the same power law decay (12) with $\alpha\simeq2.0$.

V. SUMMARY AND DISCUSSIONS

First, we have studied analytically the macroscopic failure of a homogeneous fiber bundle model where each fiber has a unique strength (σ_c). At zero noise $(\tilde{T}=0)$ all the fibers of the bundle fail simultaneously for $\sigma\geqslant\sigma_c$, while at $\tilde{T}\neq0$ each fiber has got a nonvanishing failure probability [given by Eq. (5)] due to the thermal-like activation. The dynamics of failure of the bundle has been solved using the continuum version of the recursion relation (6) for global load sharing case. The resulting expression (8) for the average failure time (τ) has qualitative features similar to that [Eq. (4)] obtained from the phenomenological nucleation rate theory applied for a Griffith's crack. Both the forms have got the desirable features that τ decreases exponentially as σ approaches σ_c from below and $\tau\simeq0$ for $\sigma\geqslant\sigma_c$. As mentioned already, although the above features agree qualitatively with the experimental observations, the precise mathematical forms we obtained here differ from the experimentally indicated forms [7]. As time t approaches τ, the fraction of unbroken fibers decays as $(\tau-t)^{\beta}$, $\beta=0_+$ and its rate of breaking grows as $(\tau-t)^{-\gamma}$, with $\gamma=1$. The avalanche size distribution $D(m)$ is also obtained analytically for the dynamics. It is seen to have a robust power law governed decay behavior $D(m)\sim m^{-\alpha}$ with $\alpha=2$. Our numerical results also confirm this behavior. Next, we have studied numerically the dynamics and the average breaking time τ for bundles where the breaking strengths are not fixed and are given by the three simple distributions $\rho(\sigma_c)$. We find that for all the three cases, the average τ fits well form (13), which is very close to the analytic form for τ in Eq. (9) for fixed failure threshold of the fibers. We have also investigated the avalanche size distributions in these models and obtained the same power law behavior, as for the fixed strength fibers.

As mentioned already, here the noise parameter [$\tilde{T}$ in Eq. (5)] cannot be identified with temperature (T in Eq. (2)], which scales with the (crack) energy. In fact, although this failure model and its dynamics are applied here to classical breakdown phenomena occurring in the fiber bundle model or (classical) percolating solids [1], they seem to be applicable to quantum breakdown due to tunneling as well. Failures in quantum percolating solids beyond their linear conducting or insulating regime, have not been studied much (see Ref. [8]). In fact, like the fuse (or dielectric breakdown) problems of percolating (or nonpercolating) systems of conductor-insulator networks, one can think of the field induced breakdown of a quantum percolating system where the phase of the system is determined through two energy scales: Fermi energy ϵ_f and the mobility edge ϵ_c. For $\epsilon_f>\epsilon_c$ the system is in conducting phase and it goes to insulating phase for $\epsilon_f<\epsilon_c$. This metal-insulator transition at $\epsilon_f=\epsilon_c$ (in higher than two-dimensional systems) and the scaling property of conductivity for $\epsilon_f>\epsilon_c$ have been studied extensively [10,11]. For the insulating phase $(\epsilon_f<\epsilon_c)$, one can have electric field induced (Zener type) breakdown (similar to dielectric breakdown of nonpercolating classical networks). This Zener breakdown of Anderson insulators or the quantum tunneling induced breakdown of impure (localized) insulators has not been studied much (see Refs. [8,12]). Unlike a (classical) fiber bundle model considered here, where all the fibers are in parallel, one can consider a dielectric composed of several elements in series having nonzero failure probability for each element due to quantum tunneling (like the noise-induced activation considered here). Any microscopic failure of such an element would result in increased field on the surviving elements and this, in turn, would enhance their failure probability. A similar dielectric failure time (τ) in such quantum or Anderson insulators is thus expected under electric field. Here σ and σ_c would be replaced by ϵ_f and ϵ_c, respectively, and $\tilde{T}$ would correspond to the inverse tunneling length determined by the electric field (with the Planck's constant as the proportionality factor, incorporating the intrinsic noise) [8].

Our study here for fatigue breakdown in the model fiber bundles shows that the average failure time for the bundle at a stress value σ less than the bundle strength σ_c (for fixed strength fibers) or $\tilde{\sigma}_c$ (for random fiber strength distributions), above which the bundle fails immediately, decreases exponentially as σ approaches σ_c or $\tilde{\sigma}_c$ from below. This has already been observed in several experiments qualitatively. We have demonstrated this fatigue behavior here both analytically and numerically for a fixed strength fiber bundle model (Sec. III) and also numerically for random fiber bundle models with nontrivial strength distributions (Sec. IV). We also believe that these observations will be useful in quantum breakdown phenomena.

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